Quantum-limited linewidth of a chaotic laser cavity

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A random-matrix theory is presented for the linewidth of a laser cavity in which the radiation is scattered chaotically. The linewidth is enhanced above the Schawlow-Townes value by the Petermann factor K, due to the non-orthogonality of the cavity modes. The factor K is expressed in terms of a non-Hermitian random matrix and its distribution is calculated exactly for the case that the cavity is coupled to the outside via a small opening. The average of K is found to depend non-analytically on the area of the opening, and to greatly exceed the most probable value.

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It has been known since the conception of the laser [1] that vacuum fluctuations of the electromagnetic field ultimately limit the narrowing of the emission spectrum by laser action. This quantum-limited linewidth, or Schawlow-Townes linewidth,

$$\delta\omega = \tfrac{1}{2}\Gamma^2/I \,, \tag{1}$$

is proportional to the square of the decay rate Γ of the lasing cavity mode [2] and inversely proportional to the output power I (in units of photons/s). Many years later it was realised [3,4] that the fundamental limit is larger than Eq. (1) by a factor K that characterises the non-orthogonality of the cavity modes. This excess noise factor, or Petermann factor, has generated an extensive literature (see the recent papers [5–9] and references therein), both because of its fundamental significance and because of its practical importance.

Theories of the enhanced linewidth usually factorise $K = K_l K_r$ into a longitudinal and transverse factor, assuming that the cavity mode is separable into longitudinal and transverse modes. Since a longitudinal or transverse mode is essentially one-dimensional, that is a major simplification. Separability breaks down if the cavity has an irregular shape or contains randomly placed scatterers. In the language of dynamical systems, one crosses over from integrable to chaotic dynamics [10]. Chaotic laser cavities have attracted much interest recently [11], but not in connection with the quantum-limited linewidth.

In this paper we present a general theory for the Petermann factor in a system with chaotic dynamics, and apply it to the simplest case of a chaotic cavity radiating through a small opening. Chaotic systems require a statistical treatment, so we compute the probability distribution of K in an ensemble of cavities with small variations in shape and size. We find that the average of $K-1$ depends *non-analytically* $\propto T \ln T^{-1}$ on the transmission probability T through the opening, so that it is beyond the reach of simple perturbation theory. The most probable value of $K-1$ is $\propto T$, hence it is parametrically smaller than the average.

The spectral statistics of chaotic systems is described by random-matrix theory [10,12]. We begin by reformulating the existing theories for the Petermann factor [8,9] in the framework of random-matrix theory. Modes of a closed cavity, in the absence of absorption or amplification, are eigenvalues of a Hermitian operator H_0. For a chaotic cavity, H_0 can be modelled by an $M \times M$ Hermitian matrix with independent Gaussian distributed elements. (The limit $M \to \infty$ at fixed spacing Δ of the modes is taken at the end of the calculation.) The matrix elements are real because of time-reversal symmetry. (This is the Gaussian orthogonal ensemble [12].) A small opening in the cavity is described by a real, nonrandom $M \times N$ coupling matrix W, with N the number of wave channels transmitted through the opening. (For an opening of area $\mathcal{A}$, $N \simeq 2\pi\mathcal{A}/\lambda^2$ at wavelength λ.) Modes of the open cavity are complex eigenvalues (with negative imaginary part) of the non-Hermitian matrix $H = H_0 - i\pi W W^T$. The scattering matrix S at frequency ω is related to H by [13]

$$S = \mathbb{1} - 2\pi i W^T(\omega - H)^{-1} W \,. \tag{2}$$

It is a unitary and symmetric, random $N \times N$ matrix, with poles at the eigenvalues of H.

We now assume that the cavity is filled with a homogeneous amplifying medium (amplification rate $1/\tau_a$). This adds a term $i/2\tau_a$ to the eigenvalues, shifting them upwards towards the real axis. The lasing mode is the eigenvalue $\Omega - i\Gamma/2$ closest to the real axis, and the laser threshold is reached when the decay rate Γ of this mode equals the amplification rate $1/\tau_a$ [14]. Near the laser threshold we need to retain only the contribution from the lasing mode (say mode number l) to the scattering matrix (2),

$$S_{nm} = -2\pi i (W^T U)_{nl}(\omega - \Omega + i\Gamma/2 - i/2\tau_a)^{-1} \cdot (U^{-1}W)_{lm} \,, \tag{3}$$

where U is the matrix of eigenvectors of H. Because H is a real symmetric matrix, we can choose U such that $U^{-1} = U^T$ and write Eq. (3) in the form

$$S_{nm} = \sigma_n \sigma_m (\omega - \Omega + i\Gamma/2 - i/2\tau_a)^{-1} , \tag{4}$$

where $\sigma_n = (-2\pi i)^{1/2} (W^T U)_{nl}$ is the complex coupling constant of the lasing mode l to the n-th wave channel.

The Petermann factor K is given by

$$\sqrt{K} = \frac{1}{\Gamma} \sum_{n=1}^{N} |\sigma_n|^2 = (U^\dagger U)_{ll} . \tag{5}$$

The second equality follows from the definition of σ_n [15], and is the matrix analogon of Siegman's non-orthogonal mode expression [4]. The first equality follows from the definition of K as the factor multiplying the Schawlow-Townes linewidth [16]. One verifies that $K \geq 1$ because $(U^\dagger U)_{ll} \geq (U^T U)_{ll} = 1$.

The relation (5) serves as the starting point for a calculation of the statistics of the Petermann factor in an ensemble of chaotic cavities. Here we restrict ourselves to the case $N = 1$ of a single wave channel, leaving the multi-channel case for future investigation. For $N = 1$ the coupling matrix W reduces to a vector $\vec{\alpha} = (W_{11}, W_{21}, \ldots, W_{M1})$. Its magnitude $|\vec{\alpha}|^2 = (M\Delta/\pi^2) w$, where $w \in [0, 1]$ is related to the transmission probability T of the single wave channel by $T = 4w(1 + w)^{-2}$. We assume a basis in which H_0 is diagonal (eigenvalues ω_q).

If the opening is much smaller than a wavelength, then a perturbation theory in $\vec{\alpha}$ seems a natural starting point. To leading order one finds

$$K = 1 + (2\pi\alpha_l)^2 \sum_{q \neq l} \frac{\alpha_q^2}{(\omega_l - \omega_q)^2} . \tag{6}$$

The frequency Ω and decay rate Γ of the lasing mode are given by ω_l and $2\pi\alpha_l^2$, respectively, to leading order in $\vec{\alpha}$. We seek the average $\langle K \rangle_{\Omega,\Gamma}$ of K for a given value of Ω and Γ. The probability to find an eigenvalue at ω_q given that there is an eigenvalue at ω_l vanishes *linearly* for small $|\omega_q - \omega_l|$, as a consequence of eigenvalue repulsion constrained by time-reversal symmetry. Since the expression (6) for K diverges *quadratically* for small $|\omega_q - \omega_l|$, we conclude that $\langle K \rangle_{\Omega,\Gamma}$ does not exist in perturbation theory. This severely complicates the problem.

We have succeeded in obtaining a finite answer for the average Petermann factor by starting from the exact relation

$$U_{ql} z_l = \omega_q U_{ql} - i\pi\alpha_q \sum_p \alpha_p U_{pl} \tag{7}$$

between the complex eigenvalues z_q of H and the real eigenvalues ω_q of H_0. Distinguishing between $q = l$ and $q \neq l$, and defining $d_q = U_{ql}/U_{ll}$, we obtain two recursion relations,

$$z_l = \omega_l - i\pi\alpha_l^2 - i\pi\alpha_l \sum_{q \neq l} \alpha_q d_q , \tag{8a}$$

$$i d_q = \frac{\pi\alpha_q}{z_l - \omega_q} \left(\alpha_l + \sum_{p \neq l} \alpha_p d_p \right) . \tag{8b}$$

The Petermann factor of the lasing mode l follows from

$$\sqrt{K} = \left(1 + \sum_{q \neq l} |d_q|^2 \right) \left| 1 + \sum_{q \neq l} d_q^2 \right|^{-1} . \tag{9}$$

We now use the fact that z_l is the eigenvalue closest to the real axis. We may therefore assume that z_l is close to the unperturbed value ω_l and replace the denominator $z_l - \omega_q$ in Eq. (8b) by $\omega_l - \omega_q$. That decouples the two recursion relations, which may then be solved in closed form,

$$z_l = \omega_l - i\pi\alpha_l^2 \, (1 + i\pi A)^{-1} , \tag{10a}$$

$$i d_q = \frac{\pi\alpha_q\alpha_l}{\omega_l - \omega_q} \, (1 + i\pi A)^{-1} . \tag{10b}$$

We have defined $A = \sum_{q \neq l} \alpha_q^2 (\omega_l - \omega_q)^{-1}$. The decay rate of the lasing mode is

$$\Gamma = -2 \, \mathrm{Im} \, z_l = 2\pi\alpha_l^2 (1 + \pi^2 A^2)^{-1} . \tag{11}$$

Since the lasing mode is close to the real axis, we may linearise the expression (9) for K with respect to Γ,

$$K = 1 + 4 \sum_{q \neq l} (\mathrm{Im} \, d_q)^2 = 1 + \frac{(2\pi\Gamma/\Delta) B}{1 + \pi^2 A^2} , \tag{12}$$

with $B = \Delta \sum_{q \neq l} \alpha_q^2 (\omega_l - \omega_q)^{-2}$.

The conditional average of K at given Γ and Ω can be written as the ratio of two unconditional averages,

$$\langle K \rangle_{\Omega,\Gamma} = 1 + (2\pi\Gamma/\Delta) \langle B (1 + \pi^2 A^2)^{-1} Z \rangle / \langle Z \rangle , \tag{13a}$$

$$Z = \delta(\Omega - \omega_l) \delta\left(\Gamma - 2\pi\alpha_l^2 (1 + \pi^2 A^2)^{-1} \right) . \tag{13b}$$

In principle one should also require that the decay rates of modes $q \neq l$ are bigger than Γ, but this extra condition becomes irrelevant for $\Gamma \to 0$. For $M \to \infty$ the distribution of α_q is Gaussian $\propto \exp(-\frac{1}{2}\alpha_q^2 \pi^2 / w\Delta)$ [12]. The average of Z over α_l yields a factor $(1 + \pi^2 A^2)^{1/2}$,

$$\langle K \rangle_{\Omega,\Gamma} = 1 + (2\pi\Gamma/\Delta) \frac{\langle B (1 + \pi^2 A^2)^{-1/2} \rangle}{\langle (1 + \pi^2 A^2)^{1/2} \rangle} , \tag{14}$$

where only the averages over α_q and ω_q ($q \neq l$) remain, at fixed $\omega_l = \Omega$.

The problem is now reduced to a calculation of the joint probability distribution $P(A, B)$. This is a technical challenge, similar to the level curvature problem of random-matrix theory [17,18]. The calculation will be presented elsewhere, here we only give the result:

$$P(A, B) = \frac{1}{6} \sqrt{\frac{\pi}{2w}} \frac{\pi^2 A^2 + w^2}{B^{7/2}} \exp\left[-\frac{w}{2B} \left(\frac{\pi^2 A^2}{w^2} + 1 \right) \right] . \tag{15}$$

Together with Eq. (14) this gives the mean Petermann factor

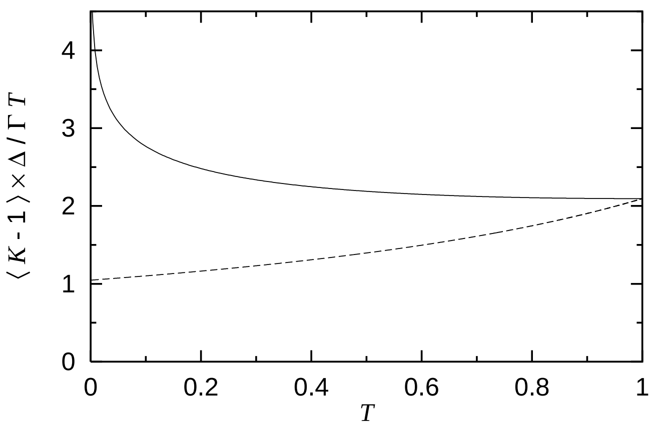

FIG. 1. Average Petermann factor K for a chaotic cavity having an opening with transmission probability T. The average is performed at fixed decay rate Γ of the lasing mode, assumed to be much smaller than the mean modal spacing Δ. The solid curve is the result (16) in the presence of time-reversal symmetry, the dashed curve is the result (20) for broken time-reversal symmetry. For small T, the solid curve diverges $\propto \ln T^{-1}$ while the dashed curve has the finite limit of $\pi/3$. For $T = 1$ both curves reach the value $2\pi/3$.

$$\langle K\rangle_{\Omega,\Gamma} = 1 - \frac{\Gamma}{\Delta}\frac{2\pi}{3}\frac{G^{22}_{22}\left(w^2\,\middle|\,\begin{matrix}0 & 0\\ -\frac{1}{2} & -\frac{1}{2}\end{matrix}\right)}{G^{22}_{22}\left(w^2\,\middle|\,\begin{matrix}-\frac{1}{2} & \frac{1}{2}\\ -1 & 0\end{matrix}\right)}\,, \tag{16}$$

in terms of the ratio of two Meijer G-functions. We have plotted the result in Fig. 1, as a function of $T = 4w(1+w)^{-2}$.

The non-analytic dependence of the average K on T (and hence on the area of the opening [19]) is a striking feature of our result. For $T \ll 1$, the average reduces to

$$\langle K\rangle_{\Omega,\Gamma} = 1 + \frac{\pi}{6}\frac{T\Gamma}{\Delta}\ln\frac{16}{T}\,. \tag{17}$$

The non-analyticity results from the relatively weak eigenvalue repulsion in the presence of time-reversal symmetry. If time-reversal symmetry is broken by a magneto-optical effect (as in Refs. [20,21]), then the stronger quadratic repulsion is sufficient to overcome the ω^{-2} divergence of perturbation theory and the average K becomes an analytic function of T. For this case, we find instead of Eq. (14) the simpler expression

$$\langle K\rangle_{\Omega,\Gamma} = 1 + (2\pi\Gamma/\Delta)\frac{\langle B\rangle}{\langle 1+\pi^2A^2\rangle}\,. \tag{18}$$

Using the joint probability distribution

$$P(A,B) = \frac{\left(\pi^2A^2+w^2\right)^2}{3wB^5}\exp\left[-\frac{w}{B}\left(\frac{\pi^2A^2}{w^2}+1\right)\right]\,, \tag{19}$$

we find the mean K,

$$\langle K\rangle_{\Omega,\Gamma} = 1 + \frac{\Gamma}{\Delta}\frac{4\pi w}{3(1+w^2)}\,, \tag{20}$$

shown dashed in Fig. 1. It is equal to $\langle K\rangle_{\Omega,\Gamma} = 1 + \frac{1}{3}\pi T\Gamma/\Delta$ for $T \ll 1$.

So far we have concentrated on the average Petermann factor, but from Eqs. (11), (12), and (15) we can compute the entire probability distribution of K at fixed Γ. We define $\kappa = (K-1)\Delta/\Gamma T$. A simple result for $P(\kappa)$ follows for $T = 1$,

$$P(\kappa) = \frac{4\pi^2}{3}\kappa^{-7/2}\exp(-\pi/\kappa)\,, \tag{21}$$

and for $T \ll 1$,

$$P(\kappa) = \frac{\pi}{12\kappa^2}\left(1+\frac{\pi}{2\kappa}\right)\exp\left(-\tfrac{1}{4}\pi/\kappa\right)\,,\quad \kappa T \lesssim 1\,. \tag{22}$$

As shown in Fig. 2, both distributions are very broad and asymmetric, with a long tail towards large κ [22]. The most probable (or modal) value of $K - 1 \simeq T\Gamma/\Delta$ is parametrically smaller than the mean value (17) for $T \ll 1$.

To check our analytical results we have also done a numerical simulation of the random-matrix model, generating a large number of random matrices H_0 and computing K from Eq. (5). As one can see from Fig. 2, the agreement with Eqs. (21) and (22) is flawless.

In conclusion, we have shown that chaotic scattering causes large statistical fluctuations in the quantum-limited linewidth of a laser cavity. We have examined in detail the case that the coupling to the cavity is via a single wave channel, but our random-matrix model applies more generally to coupling via an arbitrary number N of wave channels. We have computed exactly the distribution of the Petermann factor for $N = 1$. It remains

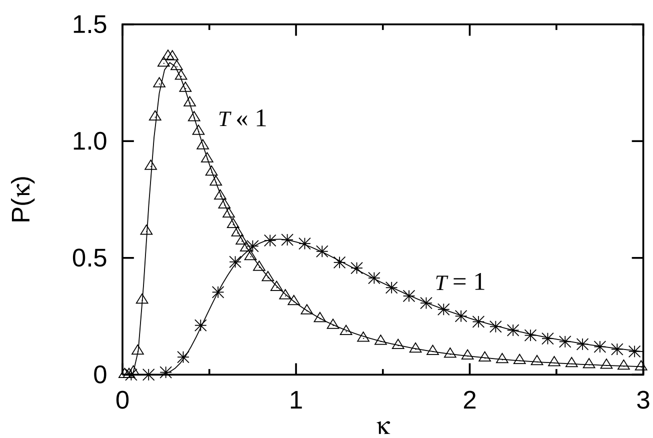

FIG. 2. Probability distribution of the rescaled Petermann factor $\kappa = (K-1)\Delta/\Gamma T$ for $T = 1$ and $T \ll 1$. The solid curves follow from Eqs. (21) and (22). The data points follow from a numerical simulation of the random-matrix model.

an open problem to do the same for $N > 1$. This problem is related to several recent studies of the statistics of eigenfunctions of non-Hermitian Hamiltonians [23,24], but is complicated by the constraint that the corresponding eigenvalue is the closest to the real axis. Our study of a system with a fully chaotic phase space complements previous theoretical work on systems with an integrable dynamics. Chaotic laser cavities of recent experimental interest [25] have a phase space that includes both integrable and chaotic regions. The study of the quantum-limited linewidth of such mixed systems is a challenging problem for future research.

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[16] For the first equality in Eq. (5), write the linewidth $\delta\omega = \Gamma - 1/\tau_a$ in terms of the output power $I = \int \mathrm{tr}\, SS^\dagger d\omega/2\pi = (\sum_n |\sigma_n|^2)^2(\Gamma - 1/\tau_a)^{-1} = K\Gamma^2/\delta\omega$. The linewidth differs from the Schawlow-Townes value (1) by a factor $2K$. The extra factor 2 arises from the suppression of amplitude fluctuations in the non-linear regime above the laser threshold, as explained by P. Goldberg, P. W. Milonni, and B. Sundaram, Phys. Rev. A **44**, 1969 (1991).

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